

含噪声的三体量子非局域共享的持久性*

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量子非局域共享问题是量子通信中的一类基本问题. 目前通过违反 Mermin 不等式和 NS 不等式证明了有无限个独立的 Charlies 可以与一对 Alice 和 Bob 共享标准三体量子非局域性和真正无信号非局域性. 然而, 上述结论是在理想状态下得出的, 在实际操作过程中不可避免地会受到各种噪声的影响, 这些因素都可能导致量子非局域性的减弱甚至消失. 本文主要针对含有噪声的三体量子共享非局域性的持久性问题进行一系列分析. 证明了即使在噪声环境下, 单个 Alice 和 Bob 仍然可以与任意多个 Charlies 共享标准三体量子非局域性的充分条件. 此外还给出了在非理想状态下, 任意多个独立的 Charlies 与一对 Alice 和 Bob 共享真正无信号非局域性的充分条件. 结果表明, 即使在非理想的条件下, 只要噪声参数满足相应的条件, 标准三体量子非局域性和真正无信号量子非局域性仍然可以在多方之间安全地共享, 这可以为实际量子通信过程提供有价值的参考.

关键词: 非局域性, 三体量子系统, 噪声

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1 引言

量子非局域性是指在量子力学中, 两个或多个量子系统之间存在的一种超越传统局域约束的相互依赖关系. 1964 年, 贝尔^[1]根据贝尔不等式的推导揭示了这一现象, 这不仅挑战了经典物理的局域性假设, 也为量子信息科学的发展奠定了理论基础. 量子非局域性在多个领域中具有重要的应用和意义, 比如在量子通信中, 它被广泛应用于量子密钥分发^[2-4], 确保信息传输的安全性, 以及量子保密通信^[5,6], 保障信息隐私; 在随机认证协议中, 量子非局域性用于确保身份验证的安全性和可靠性^[7-11]. 此外, 非局域性还被应用于与设备无关的量子信息处理认证方案, 为量子安全性提供了理论保障^[12-14]. 量子非局域性的研究为深入理解量子力学的本质

和基本原理提供了重要的线索, 从而推动了量子技术的发展.

近年来, 在参考文献^[15-18]中, 一对纠缠量子比特产生的非局域性是否可以被参与方无限共享的相关问题引起了广泛的关注. 对于两体量子系统, Silva 等^[15]引入了这种序列场景, 并通过弱测量证明了最多两个序列观察者 (Bob) 可以与单个 Alice 共享非局域性. 在近期的研究中, 通过顺序测量极大纠缠纯双量子位态的一半, 发现了使得无限数量的独立 Bobs 可以和单个 Alice 违反 CHSH 不等式的最优测量方法, 并推广到高维量子系统^[15,19,20]. 对于三体量子系统, 通过违反 Mermin 不等式^[21], 证明了最多有 6 个 Charlies 可以同时与一对 Alice 和 Bob 共享标准三体量子非局域性^[22]. 此外, 基于 Svetlichny 不等式^[23], 证明了最多 2 个 Charlies 可以同时与一对 Alice 和 Bob 共享真正无信号非

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局域^[22].

既然量子非局域性可以通过违反不同类型的不等式来观察, 那么有趣的问题是, 是否存在某种非局域性使得任意数量的 Charlies 可以同时与一对 Alice 和 Bob 共享^[24]. 在最近的研究中, 这个问题已经得到了解决. 2023 年, Xi 等^[25]通过一组最优测量证明任意多个 Charlies 可以与一对 Alice 和 Bob 共享初始态为 GHZ 态时标准三体量子非局域性和真正的无信号非局域性. 然而上述讨论总是局限于理想的场景, 即状态的产生和测量都是无噪声的. 不久前, Mukherjee 等^[26]从实际的角度研究了含噪声线性网络的非局域相关持续性问题, 其中考虑了不同的噪声来源, 如纠缠产生的误差、噪声量子信道上的通信和测量中的缺陷. 此外, Mukherjee^[27]还从实际的角度研究了含噪声的三角网络中非三域相关的容错问题. 事实上, 使用单量子位和双量子位操作时都可能产生不完美纠缠态^[28], 而且测量过程中的低检测效率也会导致实验噪声的产生^[29]. 也就是在实际的问题中噪声的产生是不可避免的, 在近期的文章中含噪声的两体量子共享持久性问题已经得以研究^[30], 接下来本文将在非理想环境下分析是否存在任意多个 Charlies 可以与单个 Alice 和 Bob 共享标准三体量子非局域性和真正无信号非局域性, 这里主要考虑初始量子纠缠态产生的白噪声以及测量产生的噪声.

本文的结构如下: 第 2 节回顾一些基本的定义; 第 3 节讨论在噪声环境下标准三体量子非局域性是否可以无限共享以及无限共享的充分条件, 并分析不同噪声参数对该非局域持久性的影响; 第 4 节讨论在非理想条件下真无信号非局域性可以无限共享的充分条件, 并对一些特殊的噪声环境下这种非局域性的持久性进行分析.

2 基本定义

本文考虑初始量子态 $\rho_{ABC^{(1)}} = |GHZ\rangle\langle GHZ|$, 其中 $|GHZ\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$. 如图 1 所示, 量子态 $\rho_{ABC^{(1)}}$ 最初分布在 Alice, Bob 和 Charlie⁽¹⁾ 之间, 这三个粒子在空间上是分离的, 并且在 Alice, Bob 和多个 Charlies 之间共享, Alice 对第一个粒子进行测量 $\mathbf{X}$, 得到结果 a ; Bob 对第二个粒子进行测量 $\mathbf{Y}$, 得到结果 b ; Charlie⁽¹⁾ 对第三个粒子进行测量 $\mathbf{Z}^{(1)}$, 得到结果 $c^{(1)}$, 测量完成后, 将该粒子传递给 Charlie⁽²⁾, 然后 Charlie⁽²⁾ 测量后将粒子传递给 Charlie⁽³⁾, 以此类推. 此外, 本文考虑的是无偏输入场景, 即每个 Charlie 的所有可能的测量设置都是等概率的, 且每个 Charlie 执行独立于此序列中前面 Charlie 的测量选择和结果的测量. 如果 Charlie^(k-1) 执行测量为 $\mathbf{C}_{c|z}^{(k-1)}$, 则 Alice, Bob 和 Charlie^(k) 之间测量后的共享量子态可用 Lüders 规则描述^[19]:

$$\rho_{ABC^{(k)}} = \frac{1}{2} \sum_{c,z} \left(\mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\mathbf{C}_{c|z}^{(k-1)}} \rho_{ABC^{(k-1)}} \mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\mathbf{C}_{c|z}^{(k-1)}} \right). \quad (1)$$

下面给出本文所考虑的标准三体量子系统非局域共享的两种形式:

1) 如果违反如下 Mermin 不等式^[21]:

$$\langle \mathbf{X}_1 \mathbf{Y}_0 \mathbf{Z}_0 \rangle + \langle \mathbf{X}_0 \mathbf{Y}_1 \mathbf{Z}_0 \rangle + \langle \mathbf{X}_0 \mathbf{Y}_0 \mathbf{Z}_1 \rangle - \langle \mathbf{X}_1 \mathbf{Y}_1 \mathbf{Z}_1 \rangle \leq 2, \quad (2)$$

那它们之间就是标准三体量子非局域的. 其中 $\langle \mathbf{X}_i \mathbf{Y}_j \mathbf{Z}_k \rangle = \sum_{abc} (-1)^{a+b+c} P(abc | \mathbf{X}_i \mathbf{Y}_j \mathbf{Z}_k)$ 表示 Alice 测量系统为 $\mathbf{X}_i$, 结果为 a ; Bob 的测量系统为 $\mathbf{Y}_j$, 结果为 b ; Charlie 的测量系统为 $\mathbf{Z}_k$, 结果

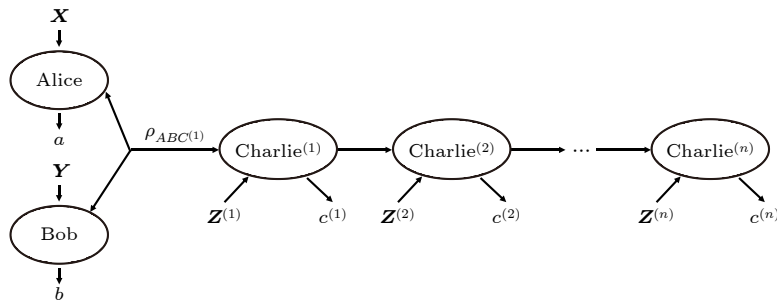


图 1 共享三体量子非局域场景

Fig. 1. Scenario of sharing the genuine tripartite nonlocality.

为 c 的联合期望值. 由玻恩法则可得:

$$P(abc|\mathbf{X}_i\mathbf{Y}_j\mathbf{Z}_k) = \text{Tr}[\rho_{ABC}(\mathbf{A}_{a|i} \otimes \mathbf{B}_{b|j} \otimes \mathbf{C}_{c|k})],$$

其中, ρ_{ABC} 表示三者之间共享的量子态, $\mathbf{A}_{a|i}$, $\mathbf{B}_{b|j}$, $\mathbf{C}_{c|k}$ 为 Alice, Bob, Charlie 所执行的测量.

由于 $\mathbf{X}_i = \mathbf{A}_{0|i} - \mathbf{A}_{1|i}$, $\mathbf{Y}_i = \mathbf{B}_{0|i} - \mathbf{B}_{1|i}$ 且 $\mathbf{Z}_i^{(k)} = \mathbf{C}_{0|i}^{(k)} - \mathbf{C}_{1|i}^{(k)}$, $i = 0, 1$, 所以 Alice, Bob 和 Charlie $^{(k)}$ 之间的 Mermin 值定义为

$$I_M^{(k)} = \text{Tr} \left[\rho_{ABC^{(k)}} \left(\mathbf{X}_1 \mathbf{Y}_0 \mathbf{Z}_0^{(k)} + \mathbf{X}_0 \mathbf{Y}_1 \mathbf{Z}_0^{(k)} + \mathbf{X}_0 \mathbf{Y}_0 \mathbf{Z}_1^{(k)} - \mathbf{X}_1 \mathbf{Y}_1 \mathbf{Z}_1^{(k)} \right) \right]. \quad (3)$$

2) 如果违反如下 NS 不等式 [24]:

$$\langle \mathbf{Y}_0 \mathbf{Z}_0 \rangle + \langle \mathbf{X}_0 \mathbf{Z}_0 \rangle + \langle \mathbf{X}_1 \mathbf{Y}_0 \rangle - \langle \mathbf{X}_0 \mathbf{Y}_1 \mathbf{Z}_1 \rangle + \langle \mathbf{X}_1 \mathbf{Y}_1 \mathbf{Z}_1 \rangle \leq 3, \quad (4)$$

那它们之间就是真正无信号非局域性的. 其中, $\langle \mathbf{X}_i \mathbf{Y}_j \mathbf{Z}_k \rangle = \sum_{abc} (-1)^{a+b+c} P(abc | \mathbf{X}_i \mathbf{Y}_j \mathbf{Z}_k)$. 因此, 同上可得 Alice, Bob 和 Charlie $^{(k)}$ 之间 NS 值可定义为

$$I_{\text{NS}}^{(k)} = \text{Tr} \left[\rho_{ABC^{(k)}} \left(\mathbf{Y}_0 \mathbf{Z}_0^{(k)} + \mathbf{X}_0 \mathbf{Z}_0^{(k)} + \mathbf{X}_1 \mathbf{Y}_0 - \mathbf{X}_0 \mathbf{Y}_1 \mathbf{Z}_1^{(k)} + \mathbf{X}_1 \mathbf{Y}_1 \mathbf{Z}_1^{(k)} \right) \right]. \quad (5)$$

3 基于 Mermin 不等式的带噪声的标准三体量子非局域共享

在参考文献 [25] 中, 引入了一组最优测量, 使得当初始态为 $|GHZ\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$ 时, 任意多个独立的 Charlies 可与单个的 Alice 和 Bob 共享标准三体量子非局域性. Alice, Bob, 以及对于 $k = 1, 2, \dots, n$ 的 Charlie $^{(k)}$ 的最优测量策略分别为以下双结果正算子值测量 (POVMs):

$$\mathbf{A}_{0|0} := \frac{\mathbf{I} + \boldsymbol{\sigma}_x}{2}, \mathbf{A}_{0|1} := \frac{\mathbf{I} + \boldsymbol{\sigma}_y}{2}; \quad (6)$$

$$\mathbf{B}_{0|0} := \frac{\mathbf{I} - \theta \boldsymbol{\sigma}_y}{2}, \mathbf{B}_{0|1} := \frac{\mathbf{I} + \theta \boldsymbol{\sigma}_x}{2}; \quad (7)$$

$$\mathbf{C}_{0|0}^{(k)} := \frac{\mathbf{I} + \boldsymbol{\sigma}_x}{2}, \mathbf{C}_{0|1}^{(k)} := \frac{\mathbf{I} + \gamma_k \boldsymbol{\sigma}_y}{2}, \quad (8)$$

其中, $\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ 为泡利矩阵,

$\theta \in (0, 1)$. 因此可得, 观测值为 $\mathbf{X}_i = \mathbf{A}_{0|i} - \mathbf{A}_{1|i}$, $\mathbf{Y}_i = \mathbf{B}_{0|i} - \mathbf{B}_{1|i}$, $\mathbf{Z}_i^{(k)} = \mathbf{C}_{0|i}^{(k)} - \mathbf{C}_{1|i}^{(k)}$, 其中, $i = 0, 1$.

基于上述结果, 下面讨论带噪声的情况, 设 $\alpha_1 \in [0, 1]$ 表征测量算子 $\{\mathbf{A}_{0|0}, \mathbf{A}_{1|0}\}$ 的不完全性, 即以 $1 - \alpha_1$ 的概率未检测到, 也就是以 α_1 为噪声参数. 同样, 测量 $\{\mathbf{A}_{0|1}, \mathbf{A}_{1|1}\}$, $\{\mathbf{B}_{0|0}, \mathbf{B}_{1|0}\}$, $\{\mathbf{B}_{0|1}, \mathbf{B}_{1|1}\}$, $\{\mathbf{C}_{0|0}^{(k)}, \mathbf{C}_{1|0}^{(k)}\}$, $\{\mathbf{C}_{0|1}^{(k)}, \mathbf{C}_{1|1}^{(k)}\}$ 的噪声参数分别为 $\alpha_2, \beta_1, \beta_2, \mu_k, \nu_k$, $k = 1, 2, \dots, n$, 其中 $\alpha_2, \beta_1, \beta_2, \mu_k, \nu_k \in [0, 1]$. 则噪声 POVMs 可以表示为

$$\begin{aligned} \bar{\mathbf{A}}_{0|0} &= \alpha_1 \mathbf{A}_{0|0} + \frac{1 - \alpha_1}{2} \mathbf{I}, \\ \bar{\mathbf{A}}_{0|1} &= \alpha_2 \mathbf{A}_{0|1} + \frac{1 - \alpha_2}{2} \mathbf{I}, \end{aligned} \quad (9)$$

$$\begin{aligned} \bar{\mathbf{B}}_{0|0} &= \beta_1 \mathbf{B}_{0|0} + \frac{1 - \beta_1}{2} \mathbf{I}, \\ \bar{\mathbf{B}}_{0|1} &= \beta_2 \mathbf{B}_{0|1} + \frac{1 - \beta_2}{2} \mathbf{I}, \end{aligned} \quad (10)$$

$$\begin{aligned} \bar{\mathbf{C}}_{0|0}^{(k)} &= \mu_k \mathbf{C}_{0|0}^{(k)} + \frac{1 - \mu_k}{2} \mathbf{I}, \\ \bar{\mathbf{C}}_{0|1}^{(k)} &= \nu_k \mathbf{C}_{0|1}^{(k)} + \frac{1 - \nu_k}{2} \mathbf{I}. \end{aligned} \quad (11)$$

基于上述讨论, 相应的噪声观测值为

$$\begin{aligned} \bar{\mathbf{X}}_0 &= \bar{\mathbf{A}}_{0|0} - \bar{\mathbf{A}}_{1|0} = \alpha_1 \mathbf{X}_0, \\ \bar{\mathbf{X}}_1 &= \bar{\mathbf{A}}_{0|1} - \bar{\mathbf{A}}_{1|1} = \alpha_2 \mathbf{X}_1; \end{aligned} \quad (12)$$

$$\begin{aligned} \bar{\mathbf{Y}}_0 &= \bar{\mathbf{B}}_{0|0} - \bar{\mathbf{B}}_{1|0} = \beta_1 \mathbf{Y}_0, \\ \bar{\mathbf{Y}}_1 &= \bar{\mathbf{B}}_{0|1} - \bar{\mathbf{B}}_{1|1} = \beta_2 \mathbf{Y}_1; \end{aligned} \quad (13)$$

$$\begin{aligned} \bar{\mathbf{Z}}_0^{(k)} &= \bar{\mathbf{C}}_{0|0}^{(k)} - \bar{\mathbf{C}}_{1|0}^{(k)} = \mu_k \mathbf{C}_0^{(k)}, \\ \bar{\mathbf{Z}}_1^{(k)} &= \bar{\mathbf{C}}_{0|1}^{(k)} - \bar{\mathbf{C}}_{1|1}^{(k)} = \nu_k \mathbf{C}_1^{(k)}. \end{aligned} \quad (14)$$

对于初始量子态 $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, 考虑如下白噪声情况, 其中 $\beta \in [0, 1]$:

$$\bar{\rho}_{ABC^{(1)}} = \beta |GHZ\rangle \langle GHZ| + \frac{(1 - \beta)}{8} \mathbf{I}_8. \quad (15)$$

定理 3.1 对于 (15) 式的带噪声初始态 $\bar{\rho}_{ABC^{(1)}}$, 假设 Alice, Bob 和 Charlies 执行带噪声测量 (6)—(8) 式及 (9)—(11) 式, 则 $\rho_{ABC^{(k)}}$ 的 Mermin 期望值 $I_M^{(k)\text{noisy}}$ 为

$$I_M^{(k)\text{noisy}} = 2^{1-k} \beta \theta \left[\mu_k (\alpha_2 \beta_1 + \alpha_1 \beta_2) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \nu_j^2 \gamma_j^2} \right) + \gamma_k \nu_k (\alpha_1 \beta_1 + \alpha_2 \beta_2) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \mu_j^2} \right) \right], \quad (16)$$

其中, $\beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_k, \nu_k \in [0, 1]$, $k = 1, 2, \dots, n$. 特别地, $I_M^{(1)\text{noisy}} = \mu_1 \beta \theta (\alpha_2 \beta_1 + \alpha_1 \beta_2) + \theta \gamma_1 \nu_1 (\alpha_1 \beta_1 + \alpha_2 \beta_2)$. 若 $I_M^{(k)\text{noisy}} > 2$, 则表明 Alice, Bob 和 Charlie^(k) 之间满足标准三体量子非局域性. 所以只需证明对于任意 $n \in N$, 存在 $\theta, \{\gamma_k\}$, 使得对所有 $k = 1, 2, \dots, n$, $I_M^{(k)\text{noisy}} > 2$ 成立即可. 由 (16) 式可得

$$I_M^{(k)\text{noisy}} > 2 \Leftrightarrow \gamma_k > \frac{2^k / \theta - \mu_k \beta (\alpha_2 \beta_1 + \alpha_1 \beta_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2})}{\nu_k \beta (\alpha_1 \beta_1 + \alpha_2 \beta_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})}, \quad (17)$$

要找到一个序列 $\{\gamma_k\}$ 使得对于所有 $k = 1, 2, \dots, n$, 满足 $\gamma_k \in [0, 1]$ 且 $I_M^{(k)\text{noisy}} > 2$ 成立. 因此对于 $\varepsilon > 0$, 令 $\gamma_1 := (1 + \varepsilon) \frac{2/\theta - \mu_1 \beta (\alpha_2 \beta_1 + \alpha_1 \beta_2)}{\nu_1 \beta (\alpha_1 \beta_1 + \alpha_2 \beta_2)}$, 当 $k \geq 2$,

$$\gamma_k := \begin{cases} (1 + \varepsilon) \frac{2^k / \theta - \mu_k \beta (\alpha_2 \beta_1 + \alpha_1 \beta_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2})}{\nu_k \beta (\alpha_1 \beta_1 + \alpha_2 \beta_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})}, & 0 \leq \gamma_{k-1} \leq 1, \\ \infty, & \text{其他.} \end{cases} \quad (18)$$

由此得出持续共享标准三体量子非局域的一个充分条件.

定理 3.2 对于 (15) 式中的含噪声初始量子态 $\bar{\rho}_{ABC^{(1)}}$, 假设 Alice, Bob 和 Charlie^(k) 进行噪声测量 (6)—(8) 式及 (9)—(11) 式, 如果存在 $\varepsilon > 0$ 使得

$$\max_{k=1, 2, \dots, n} \{x_k(\varepsilon)\} \leq 1,$$

则存在任意多个 Charlies 与单个的 Alice 和 Bob 共享标准三体量子非局域性. 其中

$$x_k(\varepsilon) = 2^k \left/ \left[\frac{\beta (\alpha_1 \beta_1 + \alpha_2 \beta_2) \nu_k \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})}{1 + \varepsilon} + \mu_k \beta (\alpha_2 \beta_1 + \alpha_1 \beta_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2}) \right] \right.$$

考虑到不同种类的噪声对非局域性持久性影响是不同的, 接下来分别讨论纠缠噪声和测量噪声对上述非局域共享的持久性的具体影响.

情形 1 考虑 $C_{0|1}^{(k)}$ 产生的测量噪声 ν_k 对共享持久性的影响, 为简便起见假设 $\beta = \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, 对于任意的 k , $\mu_k = 1$, $\nu_k = \nu$, 此时意味着 Alice, Bob 的测量噪声, Charlie^(k) 的相应测量 $C_{0|0}^{(k)}$ 产生的噪声以及初始态产生的白噪声均消失. 根据定理 3.1 得

$$I_M^{(k)\text{noisy}} = 2^{2-k} \theta \left[\prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu^2 \gamma_j^2}) + \nu \gamma_k \right],$$

由上可得, 对于 $\varepsilon > 0$, 当 $0 \leq \gamma_{k-1}(\varepsilon, \nu, \theta) \leq 1$ 时,

$$\gamma_k(\varepsilon, \nu, \theta) = (1 + \varepsilon) \frac{\frac{2^{k-1}}{\theta} - \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu^2 \gamma_j^2})}{\nu},$$

其中 $\gamma_1(\varepsilon, \nu, \theta) = (1 + \varepsilon) \frac{1/\theta - 1}{\nu}$. 根据文献 [25] 可得, 当 $\nu = 1$ 时, 所有相应的噪声消失, 那么对于正整数 n , 都可以找到一个 $\theta_n \in (0, 1)$ 使得对于所有的 $\theta \in (\theta_n, 1)$ 都有 $0 < \gamma_1(\theta) < \gamma_2(\theta) < \dots < \gamma_n(\theta) < 1$; 当 $\nu \in (0, 1)$ 时, 可根据文献 [25] 中定理 1 的证明进行类似的处理, 得到此时仍然可无限共享. 这说明该特定条件下标准三体量子非局域共享的持久性得以保持, 在实际应用中, 如果能尽可能地使除 $C_{0|1}^{(k)}$ 产生的测量噪声外的其他噪声消除, 就能有效减小因噪声导致的非局域衰减. 这种特定条件下的鲁棒性为量子信息的传输和存储提供了新的可能性.

情形 2 接下来考虑初始量子态产生的纠缠噪声 β 对共享持久性的影响, 假设 $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, 对于任意的 k , $\mu_k = \nu_k = 1$, 此时意味着所有的测量噪声均消失. 由定理 3.1 得

$$I_M^{(k)\text{noisy}} = 2^{2-k} \beta \theta \left[\prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \gamma_j^2} \right) + \gamma_k \right],$$

对于任意的 $\varepsilon > 0$, 当 $0 \leq \gamma_{k-1}(\varepsilon, \beta, \theta) \leq 1$ 时,

$$\gamma_k(\varepsilon, \beta, \theta) = (1 + \varepsilon) \left[\frac{2^{k-1}}{\theta \beta} - \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \gamma_j^2} \right) \right],$$

其中 $\gamma_1(\varepsilon, \beta, \theta) = (1 + \varepsilon) \left(\frac{1}{\theta \beta} - 1 \right)$. 存在 n 个 Charlies 可以与单个 Alice 和 Bob 共享非局域性等价于对于所有 $m \leq n$, 都有 $0 \leq \gamma_m(\varepsilon, \beta, \theta) \leq 1$, 并且当存在 l 使得 $\gamma_l(\varepsilon, \beta, \theta) > 1$ 时, 非局域性共享终止一次. 可以通过确定 $[0, 1]$ 区间内有多少个 $\gamma_k(\varepsilon, \beta, \theta)$ 来计算与 Alice 和 Bob 共享非局域的 Charlies 的最大数目 $n_{\max}$. 显然地, 对于不同的参数值 $n_{\max}$ 也是不同的. 例如, 取 $\varepsilon = 10^{-5}$, 当 $(\beta, \theta) = (0.9, 0.9)$ 时, $n_{\max} = 2$; 当 $(\beta, \theta) = (2/3, 3/4)$ 时, $n_{\max} = 0$; 当 $(\beta, \theta) = (3/4, 4/5)$ 时, $n_{\max} = 1$. 通过例子可以发现当 β 和 θ 的乘积越大时, $n_{\max}$ 越大.

4 利用 NS 不等式共享含噪声的真正无信号非局域性

对于真正的无信号非局域性, 考虑初始态 $|GHZ\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$. 并考虑如下测量 [25]:

Alice 的 POVMs 定义为

$$\begin{aligned} A_{0|0} &= \frac{I + \cos\theta \sigma_z - \sin\theta \sigma_x}{2}, \\ A_{0|1} &= \frac{I + \cos\theta \sigma_z + \sin\theta \sigma_x}{2}, \end{aligned} \quad (19)$$

其中, $\theta \in [0, \frac{\pi}{2}]$. Bob 的 POVMs 定义为

$$B_{0|0} = \frac{I + \sigma_z}{2}, \quad B_{0|1} = \frac{I + \sigma_x}{2}. \quad (20)$$

对于任意的 $k = 1, 2, \dots, n$, Charlie^(k) 的 POVMs 定义为

$$C_{0|0}^{(k)} = [I + \sigma_z]/2, \quad C_{0|1}^{(k)} = [I + \gamma_k \sigma_x]/2, \quad (21)$$

其中,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

可观测值为 $X_i = A_{0|i} - A_{1|i}$, $Y_i = B_{0|i} - B_{1|i}$ 且 $Z_i^{(k)} = C_{0|i}^{(k)} - C_{1|i}^{(k)}$, 其中 $i = 0, 1$. 加入与第 3 节相同的噪声, 可以得到以下结论.

定理 4.1 对于 (15) 式中的带噪声初始状态 $\bar{\rho}_{ABC^{(1)}}$, 假设 Alice, Bob 和 Charlies 执行 (19)—(21) 式及 (9)—(11) 式中的噪声测量, 则 $\rho_{ABC^{(k)}}$ 的 NS 值 $I_{NS}^{(k)\text{noisy}}$ 为

$$\begin{aligned} I_{NS}^{(k)\text{noisy}} &= \beta 2^{1-k} \left[\mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \nu_j^2 \gamma_j^2} \right) + \nu_k \gamma_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \mu_j^2} \right) \right] \\ &\quad + \alpha_2 \beta_1 \beta \cos \theta, \end{aligned} \quad (22)$$

其中, $\theta \in [0, \pi/2]$, $\beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_k, \nu_k \in [0, 1]$, $k = 1, 2, \dots, n$. 特别地, $I_{NS}^{(1)\text{noisy}} = \beta [\mu_1 (\beta_1 + \alpha_1 \cos \theta + \nu_1 \gamma_1 \beta_2 \sin \theta (\alpha_1 + \alpha_2))] + \alpha_2 \beta_1 \cos \theta$.

不等式 $I_{NS}^{(k)\text{noisy}} > 3$ 意味着 Alice, Bob 和 Charlie^(k) 遵循真正的无信号非局域性. 根据 (22) 式得

$$I_{NS}^{(k)\text{noisy}} > 3 \Leftrightarrow \gamma_k > \frac{3 - \alpha_2 \beta_1 \beta \cos \theta - \beta 2^{1-k} \mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \nu_j^2 \gamma_j^2} \right)}{\beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \mu_j^2} \right)}, \quad (23)$$

定义序列 $\{\gamma_k\}$, 对于 $\varepsilon > 0$, 令

$$\gamma_k := \begin{cases} (1 + \varepsilon) \frac{3 - \alpha_2 \beta_1 \cos(\theta) \beta - \beta 2^{1-k} \mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \nu_j^2 \gamma_j^2} \right)}{\beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \mu_j^2} \right)}, & 0 \leq \gamma_{k-1} \leq 1, \\ \infty, & \text{其他,} \end{cases} \quad (24)$$

其中

$$\gamma_1 = (1 + \varepsilon) \frac{3 - \beta \cos \theta - \beta \mu_1 (\beta_1 + \alpha_1 \cos \theta)}{\beta \nu_1 \beta_2 \sin \theta (\alpha_1 + \alpha_2)},$$

这样定义的目的是使得对于所有 $k = 1, 2, \dots, n$, 满足 $\gamma_k \in [0, 1]$ 且 $I_{\text{NS}}^{(k)\text{noisy}} > 3$ 成立. 由此得出持续共享真正无信号非局域性的一个充分条件.

定理 4.2 对于 (15) 式中的带噪声初始状态 $\bar{\rho}_{ABC^{(1)}}$, 假设 Alice, Bob 和 Charlies 执行 (19)—

(21) 式及 (9)—(11) 式中的噪声测量, 如果存在某个 $\varepsilon > 0$ 使得

$$\max_{k=1,2,\dots,n} \arctan[x_{k1}(\varepsilon)] \leq \min_{k=1,2,\dots,n} \arctan[x_{k2}(\varepsilon)], \quad (25)$$

则存在 n 个 Charlies 与单个 Alice 和 Bob 共享真正无信号非局域性. 其中, 对于任意的 $k = 1, 2, \dots, n$,

$$x_{k1}(\varepsilon) = \frac{(1+\varepsilon)(\beta\alpha_2\beta_1+\alpha_1a_k)b_k - \sqrt{[(1+\varepsilon)(\beta\alpha_2\beta_1+\alpha_1a_k)b_k]^2 - [(1+\varepsilon)^2(3-\beta_1a_k)^2 - b_k^2](1+\varepsilon)^2(3-\beta_1a_k)^2 - (1+\varepsilon)^2(\beta\alpha_2\beta_1+\alpha_1a_k)^2}}{(1+\varepsilon)^2(3-\beta_1a_k)^2 - b_k^2},$$

$$x_{k2}(\varepsilon) = \frac{(1+\varepsilon)(\beta\alpha_2\beta_1+\alpha_1a_k)b_k + \sqrt{[(1+\varepsilon)(\beta\alpha_2\beta_1+\alpha_1a_k)b_k]^2 - [(1+\varepsilon)^2(3-\beta_1a_k)^2 - b_k^2](1+\varepsilon)^2(3-\beta_1a_k)^2 - (1+\varepsilon)^2(\beta\alpha_2\beta_1+\alpha_1a_k)^2}}{(1+\varepsilon)^2(3-\beta_1a_k)^2 - b_k^2},$$

且

$$a_k = \beta 2^{1-k} \mu_k \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \nu_j^2}\right), \quad b_k = \beta 2^{1-k} \nu_k \beta_2 (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \mu_j^2}\right).$$

由此可得, 若对于任意的 $\varepsilon > 0$, 都有 $\arctan[x_{n+1,1}(\varepsilon)] > \min_{k=1,2,\dots,n} \arctan[x_{k2}(\varepsilon)]$, 那么, Charlie⁽ⁿ⁺¹⁾ 不能与单独的 Alice 和 Bob 共享初始态为 $\bar{\rho}_{ABC^{(1)}}$ 这种非局域.

特别地, 考虑测量 $C_{0|1}^{(k)}$ 产生的噪声 ν_k 和测量 $B_{0|1}$ 产生的噪声 β_2 对该非局域共享持久性的影响, 假设定理 4.1 中 $\alpha_1 = \alpha_2 = \beta_1 = \beta = 1$, 且对于任意的 k , $\mu_k = 1$, $\nu_k = \nu$, 这意味着除测量 $C_{0|1}^{(k)}$, $B_{0|1}$ 产生的噪声外, 其他噪声均消失, 由此可得

$$I_{\text{NS}}^{(k)\text{noisy}} = 2^{1-k} \left[(1 + \cos \theta) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \nu^2 \gamma_j^2}\right) + 2\nu \gamma_k \beta_2 \sin \theta \right] + \cos \theta,$$

所以此时

$$\gamma_k(\theta, \nu, \beta_2) := \begin{cases} (1+\varepsilon) \frac{1 - \cos \theta}{\nu \beta_2 \sin \theta}, & k = 1, \\ (1+\varepsilon) \frac{3 - \cos \theta - 2^{1-k} (1 + \cos \theta) \prod_{j=1}^{k-1} \left(1 + \sqrt{1 - \nu^2 \gamma_j^2}\right)}{2^{2-k} \nu \beta_2 \sin \theta}, & 0 \leq \gamma_{k-1}(\theta, \nu, \beta_2) \leq 1, \\ \infty, & \text{其他.} \end{cases} \quad (26)$$

由于 $\frac{\gamma_k(\theta, \nu, \beta_2)}{\gamma_{k-1}(\theta, \nu, \beta_2)} > 2 \Leftrightarrow 0 < \gamma_{k-1}(\theta, \nu, \beta_2) \leq 1$ 且若 $i = 1, 2, \dots, k-1$, $\gamma_i(\theta, \nu, \beta_2) \in (0, 1)$, 则有 $\lim_{\theta \rightarrow 0^+} \gamma_k(\theta, \nu, \beta_2) = 0$, 由参考文献 [25] 定理 2 的证明得, 仍存在任意多个 Charlies 与 Alice 和 Bob 共享真正无信号非局域性. 此时真正无信号非局域共享持久性得以保持.

5 结 论

本文主要探讨了在非理想环境中, 对于初始量子态 GHZ 态是否仍能使得无限数量的 Charlies

与 Alice 和 Bob 共享标准的三体量子非局域性以及真正的无信号非局域性. 在按顺序共享三体量子非局域性并且 Charlies 的测量为无偏输入的场景下, 本文重点考虑两种噪声因素: 测量噪声和量子本身产生的白噪声. 通过分析计算, 本文分别推导出在噪声条件下 Alice, Bob 及多个 Charlies 之间依然能够无限次共享三体量子非局域性和真正无信号非局域性的充分条件. 特别地, 发现在某些特定的噪声环境下 Alice 和 Bob 仍能和无限数量的 Charlies 实现该共享.

由此可见噪声的增加通常会降低系统的纠缠度, 从而限制与 Alice 和 Bob 共享 Charlies 的数

量, 但在如果这些噪声参数满足一定条件或者在某些特定的噪声环境下, 依然可以实现无限次共享. 这种特定条件下的鲁棒性为量子信息的传输和存储提供了新的可能性. 由于本文仅考虑了 Charlies

的测量设置是等概率的, 那么对于非等概率的情况或者初始态的噪声不是白噪声而是其他噪声时结论是否依然成立或者会有怎样的结论也值得深入分析.

附录A 相关定理的证明

定理 3.1 证明 通过 $\bar{\mathbf{X}}_0 = \bar{\mathbf{A}}_{0|0} - \bar{\mathbf{A}}_{1|0} = \alpha_1 \mathbf{X}_0$, $\bar{\mathbf{X}}_1 = \bar{\mathbf{A}}_{0|1} - \bar{\mathbf{A}}_{1|1} = \alpha_2 \mathbf{X}_1$; $\bar{\mathbf{Y}}_0 = \bar{\mathbf{B}}_{0|0} - \bar{\mathbf{B}}_{1|0} = \beta_1 \mathbf{Y}_0$, $\bar{\mathbf{Y}}_1 = \bar{\mathbf{B}}_{0|1} - \bar{\mathbf{B}}_{1|1} = \beta_2 \mathbf{Y}_1$; $\bar{\mathbf{Z}}_0^{(k)} = \bar{\mathbf{C}}_{0|0}^{(k)} - \bar{\mathbf{C}}_{1|0}^{(k)} = \mu_k \mathbf{Z}_0^{(k)}$, $\bar{\mathbf{Z}}_1^{(k)} = \bar{\mathbf{C}}_{0|1}^{(k)} - \bar{\mathbf{C}}_{1|1}^{(k)} = \nu_k \mathbf{Z}_1^{(k)}$, 可以得到

$$\begin{aligned} I_M^{(k)\text{noisy}} &= \text{Tr} \left[\rho_{ABC^{(k)}} (\bar{\mathbf{X}}_1 \bar{\mathbf{Y}}_0 \bar{\mathbf{Z}}_0^{(k)} + \bar{\mathbf{X}}_0 \bar{\mathbf{Y}}_1 \bar{\mathbf{Z}}_0^{(k)} + \bar{\mathbf{X}}_0 \bar{\mathbf{Y}}_0 \bar{\mathbf{Z}}_1^{(k)} - \bar{\mathbf{X}}_1 \bar{\mathbf{Y}}_1 \bar{\mathbf{Z}}_1^{(k)}) \right] \\ &= -\theta \alpha_2 \beta_1 \mu_k \text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x)] + \theta \alpha_1 \beta_2 \mu_k \text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x)] \\ &\quad - \theta \gamma_k \alpha_1 \beta_1 \nu_k \text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)] - \theta \gamma_k \alpha_2 \beta_2 \nu_k \text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y)], \end{aligned} \quad (\text{A1})$$

通过 Lüders 法则可以得到

$$\begin{aligned} \rho_{ABC^{(k)}} &= \frac{1}{2} \left[\left(\mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{0|0}^{(k-1)}} \right) \rho_{ABC^{(k-1)}} \left(\mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{0|0}^{(k-1)}} \right) + \left(\mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{1|0}^{(k-1)}} \right) \rho_{ABC^{(k-1)}} \left(\mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{1|0}^{(k-1)}} \right) \right. \\ &\quad \left. + \mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{0|1}^{(k-1)}} \rho_{ABC^{(k-1)}} \mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{0|1}^{(k-1)}} + \mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{1|1}^{(k-1)}} \rho_{ABC^{(k-1)}} \mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathbf{C}}_{1|1}^{(k-1)}} \right] \\ &= \left[\frac{1}{2} + \frac{1}{4} \left(\sqrt{1 - \mu_{k-1}^2} + \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2} \right) \right] \rho_{ABC^{(k-1)}} + \frac{1 - \sqrt{1 - \mu_{k-1}^2}}{4} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_1) \rho_{ABC^{(k-1)}} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_1) \\ &\quad + \frac{1 - \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2}}{4} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_2) \rho_{ABC^{(k-1)}} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_2), \end{aligned} \quad (\text{A2})$$

其中, $\sqrt{\bar{\mathbf{C}}_{c|z}^{(k-1)}}$ 的计算利用了如下公式:

$$\sqrt{\frac{\mathbf{I} \pm \gamma_k \boldsymbol{\sigma}_i}{2}} = \frac{(\sqrt{1 + \gamma_k} + \sqrt{1 - \gamma_k}) \mathbf{I} \pm (\sqrt{1 + \gamma_k} - \sqrt{1 - \gamma_k}) \boldsymbol{\sigma}_i}{2\sqrt{2}}. \quad (\text{A3})$$

然后, 通过 (A2) 式可得

$$\begin{aligned} \text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x)] &= \frac{2 + \left(\sqrt{1 - \mu_{k-1}^2} + \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2} \right)}{4} \\ &\quad \times \text{Tr}[\rho_{ABC^{(k-1)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x)] + \frac{1 - \sqrt{1 - \mu_{k-1}^2}}{4} \text{Tr}[(\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_x) \rho_{ABC^{(k-1)}} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_x) (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_1)] \\ &\quad + \frac{1 - \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2}}{4} \text{Tr}[(\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_y) \rho_{ABC^{(k-1)}} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_y) (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x)] \\ &= \frac{1 + \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2}}{2} \text{Tr}[\rho_{ABC^{(k-1)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x)] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \nu_j^2 \gamma_j^2}}{2} \text{Tr}[\bar{\rho}_{ABC^{(1)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x)]. \end{aligned} \quad (\text{A4})$$

与上面类似可得

$$\text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x)] = \frac{1 + \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2}}{2} \text{Tr}[\rho_{ABC^{(k-1)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x)] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \nu_j^2 \gamma_j^2}}{2} \text{Tr}[\bar{\rho}_{ABC^{(1)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x)], \quad (\text{A5})$$

$$\text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)] = \frac{1 + \sqrt{1 - \mu_{k-1}^2}}{2} \text{Tr}[\rho_{ABC^{(k-1)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \mu_j^2}}{2} \text{Tr}[\bar{\rho}_{ABC^{(1)}} (\boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)], \quad (\text{A6})$$

$$\text{Tr}[\rho_{ABC^{(k)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y)] = \frac{1 + \sqrt{1 - \mu_{k-1}^2}}{2} \text{Tr}[\rho_{ABC^{(k-1)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y)] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \mu_j^2}}{2} \text{Tr}[\bar{\rho}_{ABC^{(1)}} (\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_y)]. \quad (\text{A7})$$

通过 $\bar{\rho}_{ABC^{(1)}} = \beta |GHZ\rangle\langle GHZ| + \frac{(1-\beta)}{8} \mathbf{I}_8$ 可得

$$\begin{aligned}
 \text{Tr}[\bar{\rho}_{ABC^{(1)}}(\sigma_y \otimes \sigma_y \otimes \sigma_x)] &= \text{Tr}\left[\left(\beta|GHZ\rangle\langle GHZ| + \frac{(1-\beta)}{8}I_8\right)(\sigma_y \otimes \sigma_y \otimes \sigma_x)\right] \\
 &= \text{Tr}\left[\beta\left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right)\left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right)(\sigma_y \otimes \sigma_y \otimes \sigma_x) + \frac{(1-\beta)}{8}I_8(\sigma_y \otimes \sigma_y \otimes \sigma_x)\right] \\
 &= \frac{1}{2}\beta\text{Tr}[(|000\rangle + |111\rangle)(\sigma_y \otimes \sigma_y \otimes \sigma_x)(|000\rangle + |111\rangle)] + \text{Tr}\left[\frac{(1-\beta)}{8}I_8(\sigma_y \otimes \sigma_y \otimes \sigma_x)\right] = -\beta.
 \end{aligned} \tag{A8}$$

同上可得

$$\text{Tr}[\bar{\rho}_{ABC^{(1)}}(\sigma_x \otimes \sigma_x \otimes \sigma_x)] = \beta, \tag{A9}$$

$$\text{Tr}[\bar{\rho}_{ABC^{(1)}}(\sigma_x \otimes \sigma_y \otimes \sigma_y)] = -\beta, \tag{A10}$$

$$\text{Tr}[\bar{\rho}_{ABC^{(1)}}(\sigma_y \otimes \sigma_x \otimes \sigma_y)] = -\beta, \tag{A11}$$

根据 (A4)–(A7) 式以及 (A8)–(A11) 式, 可以得到

$$\begin{aligned}
 I_M^{(k)\text{noisy}} &= -\theta\alpha_2\beta_1\mu_k\text{Tr}[\rho_{ABC^{(k)}}(\sigma_y \otimes \sigma_y \otimes \sigma_x)] + \theta\alpha_1\beta_2\mu_k\text{Tr}[\rho_{ABC^{(k)}}(\sigma_x \otimes \sigma_x \otimes \sigma_x)] \\
 &\quad - \theta\gamma_k\alpha_1\beta_1\nu_k\text{Tr}[\rho_{ABC^{(k)}}(\sigma_x \otimes \sigma_y \otimes \sigma_y)] - \theta\gamma_k\alpha_2\beta_2\nu_k\text{Tr}[\rho_{ABC^{(k)}}(\sigma_y \otimes \sigma_x \otimes \sigma_y)] \\
 &= 2^{1-k}\beta\theta\left[\mu_k(\alpha_2\beta_1 + \alpha_1\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2\gamma_j^2}) + \gamma_k\nu_k(\alpha_1\beta_1 + \alpha_2\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2})\right].
 \end{aligned} \tag{A12}$$

定理 3.2 证明 由于 $I_M^{(k)\text{noisy}} > 2$ 等价于

$$\gamma_k > \frac{2^k/\theta - \mu_k\beta(\alpha_2\beta_1 + \alpha_1\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2\gamma_j^2})}{\nu_k\beta(\alpha_1\beta_1 + \alpha_2\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2})}, \tag{A13}$$

令

$$F(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1}) = \frac{2^k/\theta - \mu_k\beta(\alpha_2\beta_1 + \alpha_1\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2\gamma_j^2})}{\nu_k\beta(\alpha_1\beta_1 + \alpha_2\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2})},$$

注意到 $F(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1}) > 0$, 那么 (A13) 式可以写为

$$\frac{\gamma_k}{F(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1})} > 1,$$

由此得到, $\inf \frac{\gamma_k}{F(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1})} > 1$, 这表明存在 $\varepsilon > 0$, 使得 $\gamma_k(\theta) = (1 + \varepsilon)F(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1})$. 接下来, 对于 $\varepsilon > 0$, 定义如下的 $\{\gamma_k\}$, 对于 $k \geq 2$,

$$\gamma_k(\theta) := \begin{cases} (1 + \varepsilon) \frac{2^k/\theta - \mu_k\beta(\alpha_2\beta_1 + \alpha_1\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2\gamma_j^2})}{\nu_k\beta(\alpha_1\beta_1 + \alpha_2\beta_2)\prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2})}, & 0 \leq \gamma_{k-1}(\theta) \leq 1, \\ \infty, & \text{其他,} \end{cases} \tag{A14}$$

其中, $\gamma_1(\theta) = (1 + \varepsilon) \frac{2/\theta - \mu_1\beta(\alpha_2\beta_1 + \alpha_1\beta_2)}{\nu_1\beta(\alpha_1\beta_1 + \alpha_2\beta_2)}$. 显然, 对于 $k \geq 2$ 以及任意的 $j = 1, 2, \dots, \gamma_j > 0$. 为使得 $\gamma_k \leq 1$, 有

$$(1 + \varepsilon) \left[\frac{2^k}{\theta} - \mu_k\beta(\alpha_2\beta_1 + \alpha_1\beta_2) \prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2\gamma_j^2}) \right] \leq \nu_k\beta(\alpha_1\beta_1 + \alpha_2\beta_2) \prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2}). \tag{A15}$$

如果对于任意的 $i = 1, 2, \dots, k-1$, $\gamma_i(\theta) \leq 1$, 那么

$$\prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2\gamma_j^2}(\theta)) \geq \prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2}). \tag{A16}$$

所以当

$$(1 + \varepsilon) \left[\frac{2^k}{\theta} - \mu_k\beta(\alpha_2\beta_1 + \alpha_1\beta_2) \prod_{j=1}^{k-1}(1 + \sqrt{1 - \nu_j^2}) \right] \leq \nu_k\beta(\alpha_1\beta_1 + \alpha_2\beta_2) \prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2}), \tag{A17}$$

有 $\gamma_k \leq 1$. 因此,

$$\theta \geq 2^k \left/ \left[\frac{\beta(\alpha_1\beta_1 + \alpha_2\beta_2)\nu_k \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})}{1 + \varepsilon} + \mu_k\beta(\alpha_2\beta_1 + \alpha_1\beta_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2}) \right] \right. \quad (\text{A18})$$

对于任意的 $k = 1, 2, \dots, n$, 设

$$x_k(\varepsilon) = 2^k \left/ \left[\frac{\beta(\alpha_1\beta_1 + \alpha_2\beta_2)\nu_k \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})}{1 + \varepsilon} + \mu_k\beta(\alpha_2\beta_1 + \alpha_1\beta_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2}) \right] \right.,$$

注意到对于 $k=1, 2, \dots, n$, $x_k(\varepsilon) > 0$. 所以如果 $\max_{k=1,2,\dots,n} \{x_k(\varepsilon)\} \leq 1$, 存在 $\theta \in \left[\max_{k=1,2,\dots,n} \{x_k(\varepsilon)\}, 1 \right)$ 使得对于任意的 $k=1, 2, \dots, n$, $\gamma_k(\theta) \leq 1$ 成立.

定理 4.1 证明 由 $\bar{\mathbf{X}}_0 = \alpha_1 \mathbf{X}_0$, $\bar{\mathbf{X}}_1 = \alpha_2 \mathbf{X}_1$; $\bar{\mathbf{Y}}_0 = \beta_1 \mathbf{Y}_0$, $\bar{\mathbf{Y}}_1 = \beta_2 \mathbf{Y}_1$; $\bar{\mathbf{Z}}_0^{(k)} = \mu_k \mathbf{Z}_0^{(k)}$, $\bar{\mathbf{Z}}_1^{(k)} = \nu_k \mathbf{Z}_1^{(k)}$ 得

$$\begin{aligned} I_{\text{NS}}^{(k)\text{noisy}} &= \text{Tr} \left[\rho_{ABC^{(k)}} (\bar{\mathbf{Y}}_0 \bar{\mathbf{Z}}_0^{(k)} + \bar{\mathbf{X}}_0 \bar{\mathbf{Z}}_0^{(k)} + \bar{\mathbf{X}}_1 \bar{\mathbf{Y}}_0 - \bar{\mathbf{X}}_0 \bar{\mathbf{Y}}_1 \bar{\mathbf{Z}}_1^{(k)} + \bar{\mathbf{X}}_1 \bar{\mathbf{Y}}_1 \bar{\mathbf{Z}}_1^{(k)}) \right] \\ &= \beta_1 \mu_k \text{Tr} [\rho_{ABC^{(k)}} \mathbf{I} \otimes \boldsymbol{\sigma}_z \otimes \boldsymbol{\sigma}_z] + \alpha_1 \mu_k \text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z - \sin \theta \boldsymbol{\sigma}_x) \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_z] \\ &\quad + \alpha_2 \beta_1 \text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z + \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_z \otimes \mathbf{I}] \\ &\quad - \alpha_1 \beta_2 \nu_k \gamma_k \text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z - \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x] \\ &\quad + \alpha_2 \beta_2 \nu_k \gamma_k \text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z + \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x], \end{aligned} \quad (\text{A19})$$

通过使用 Lüders 法则得到

$$\begin{aligned} \rho_{ABC^{(k)}} &= \frac{1}{2} \sum_{c,z} \left(\mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathcal{C}}_{c|z}^{(k-1)}} \rho_{ABC^{(k-1)}} \mathbf{I} \otimes \mathbf{I} \otimes \sqrt{\bar{\mathcal{C}}_{c|z}^{(k-1)}} \right) = \left[\frac{1}{2} + \frac{1}{4} (\sqrt{1 - \mu_{k-1}^2} + \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2}) \right] \rho_{ABC^{(k-1)}} \\ &\quad + \frac{1 - \sqrt{1 - \mu_{k-1}^2}}{4} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_z) \rho_{ABC^{(k-1)}} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_z) + \frac{1 - \sqrt{1 - \nu_{k-1}^2 \gamma_{k-1}^2}}{4} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_x) \rho_{ABC^{(k-1)}} (\mathbf{I} \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_x), \end{aligned} \quad (\text{A20})$$

其中, $\sqrt{\bar{\mathcal{C}}_{c|z}^{(k-1)}}$ 的计算利用了如下公式:

$$\sqrt{\frac{\mathbf{I} \pm \gamma_k \boldsymbol{\sigma}_i}{2}} = \frac{(\sqrt{1 + \gamma_k} + \sqrt{1 - \gamma_k}) \mathbf{I} \pm (\sqrt{1 + \gamma_k} - \sqrt{1 - \gamma_k}) \boldsymbol{\sigma}_i}{2\sqrt{2}}. \quad (\text{A21})$$

然后, 由 (A20) 式得到

$$\text{Tr} [\rho_{ABC^{(k)}} \mathbf{I} \otimes \boldsymbol{\sigma}_z \otimes \boldsymbol{\sigma}_z] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \nu_j^2 \gamma_j^2}}{2} \text{Tr} [\bar{\rho}_{ABC^{(1)}} \mathbf{I} \otimes \boldsymbol{\sigma}_z \otimes \boldsymbol{\sigma}_z] = \beta \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \nu_j^2 \gamma_j^2}}{2}, \quad (\text{A22})$$

$$\text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z - \sin \theta \boldsymbol{\sigma}_x) \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_z] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \nu_j^2 \gamma_j^2}}{2} \text{Tr} [\bar{\rho}_{ABC^{(1)}} (\cos \theta \boldsymbol{\sigma}_z - \sin \theta \boldsymbol{\sigma}_x) \otimes \mathbf{I} \otimes \boldsymbol{\sigma}_z] = \cos \theta \beta \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \nu_j^2 \gamma_j^2}}{2}, \quad (\text{A23})$$

$$\text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z + \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_z \otimes \mathbf{I}] = \text{Tr} [\bar{\rho}_{ABC^{(1)}} (\cos \theta \boldsymbol{\sigma}_z + \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_z \otimes \mathbf{I}] = \cos \theta \beta, \quad (\text{A24})$$

$$\text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z - \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \mu_{k-1}^2}}{2} \text{Tr} [\bar{\rho}_{ABC^{(1)}} (\cos \theta \boldsymbol{\sigma}_z - \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x] = -\sin \theta \beta \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \mu_j^2}}{2}, \quad (\text{A25})$$

$$\text{Tr} [\rho_{ABC^{(k)}} (\cos \theta \boldsymbol{\sigma}_z + \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x] = \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \mu_{k-1}^2}}{2} \text{Tr} [\bar{\rho}_{ABC^{(1)}} (\cos \theta \boldsymbol{\sigma}_z + \sin \theta \boldsymbol{\sigma}_x) \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x] = \sin \theta \beta \prod_{j=1}^{k-1} \frac{1 + \sqrt{1 - \mu_j^2}}{2}. \quad (\text{A26})$$

通过 (A19) 式和 (A22)–(A26) 式可得

$$I_{\text{NS}}^{(k)\text{noisy}} = \beta 2^{1-k} \left[\mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2}) + \nu_k \gamma_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2}) \right] + \alpha_2 \beta_1 \cos \theta \beta.$$

定理 4.2 证明 由于 $I_{\text{NS}}^{(k)\text{noisy}} > 3$ 等价于

$$\gamma_k > \frac{3 - \alpha_2 \beta_1 \cos \theta \beta - \beta 2^{1-k} \mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2})}{\beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})}, \quad (\text{A27})$$

令

$$f(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1}) = \frac{3 - \alpha_2 \beta_1 \cos \theta \beta - \beta 2^{1-k} \mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2})}{\beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})},$$

显然, $f(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1}) > 0$, 则 (A27) 式可以写成

$$\frac{\gamma_k}{f(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1})} > 1,$$

因此存在某些 $\varepsilon > 0$, 使得 $\gamma_k(\theta) = (1 + \varepsilon)f(\theta, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k, \gamma_1, \dots, \gamma_{k-1})$. 接下来, 对于 $\varepsilon > 0$, 可以定义如下的 $\{\gamma_k(\theta)\}$, 对于 $k \geq 2$,

$$\gamma_k(\theta) := \begin{cases} (1 + \varepsilon) \frac{3 - \alpha_2 \beta_1 \cos \theta \beta - \beta 2^{1-k} \mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2})}{\beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})}, & 0 \leq \gamma_{k-1} \leq 1, \\ \infty, & \text{其他,} \end{cases} \quad (\text{A28})$$

其中, $\gamma_1(\theta) = (1 + \varepsilon) \frac{3 - \alpha_2 \beta_1 \cos \theta \beta - \beta \mu_1 (\beta_1 + \alpha_1 \cos \theta)}{\beta \nu_1 \beta_2 \sin \theta (\alpha_1 + \alpha_2)}$. 显然, 对于 $\varepsilon > 0$ 和任意的 $j = 1, 2, \dots$, $\gamma_j(\theta) > 0$. 为了让 $\gamma_k(\theta) < 1$, 等价于 $(1 + \varepsilon) \left[3 - \alpha_2 \beta_1 \cos \theta \beta - \beta 2^{1-k} \mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2}) \right] < \beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})$. 如果对于 $i = 1, 2, \dots, k-1$, $\gamma_i(\theta) < 1$, 那么,

$$\prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2 \gamma_j^2(\theta)}) > \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2}), \quad (\text{A29})$$

所以当

$$(1 + \varepsilon) \left[3 - \alpha_2 \beta_1 \cos \theta \beta - \beta 2^{1-k} \mu_k (\beta_1 + \alpha_1 \cos \theta) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2}) \right] < \beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2}),$$

有 $\gamma_k(\theta) < 1$. 也就是

$$(1 + \varepsilon) \left[3 - \beta 2^{1-k} \mu_k \beta_1 \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2}) \right] - (1 + \varepsilon) \cos \theta \beta \left[\alpha_2 \beta_1 + 2^{1-k} \mu_k \alpha_1 \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2}) \right] < \beta 2^{1-k} \nu_k \beta_2 \sin \theta (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2}).$$

令 $a_k = \beta 2^{1-k} \mu_k \prod_{j=1}^{k-1} (1 + \sqrt{1 - \nu_j^2})$ 且 $b_k = \beta 2^{1-k} \nu_k \beta_2 (\alpha_1 + \alpha_2) \prod_{j=1}^{k-1} (1 + \sqrt{1 - \mu_j^2})$, 则 (A27) 式可写为

$$(1 + \varepsilon)(3 - \beta_1 a_k) - (1 + \varepsilon) \cos \theta (\beta \alpha_2 \beta_1 + \alpha_1 a_k) < \sin \theta b_k, \quad (\text{A30})$$

因此, 可得到

$$[(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2] \tan^2 \theta - 2(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k \tan \theta + (1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - (1 + \varepsilon)^2 (\beta \alpha_2 \beta_1 + \alpha_1 a_k)^2 < 0.$$

所以对任意的 $k = 1, 2, \dots, n$,

$\tan \theta_k \in$

$$\left(\frac{(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k - \sqrt{[(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k]^2 - [(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2][(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - (1 + \varepsilon)^2 (\beta \alpha_2 \beta_1 + \alpha_1 a_k)^2]}}{[(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2]}, \right. \\ \left. \frac{(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k + \sqrt{[(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k]^2 - [(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2][(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - (1 + \varepsilon)^2 (\beta \alpha_2 \beta_1 + \alpha_1 a_k)^2]}}{[(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2]} \right),$$

对于任意的 $k = 1, 2, \dots, n$, 令

$$x_{k1}(\varepsilon) = \frac{(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k - \sqrt{[(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k]^2 - [(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2][(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - (1 + \varepsilon)^2 (\beta \alpha_2 \beta_1 + \alpha_1 a_k)^2]}}{[(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2]}, \\ x_{k2}(\varepsilon) = \frac{(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k + \sqrt{[(1 + \varepsilon)(\beta \alpha_2 \beta_1 + \alpha_1 a_k) b_k]^2 - [(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2][(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - (1 + \varepsilon)^2 (\beta \alpha_2 \beta_1 + \alpha_1 a_k)^2]}}{[(1 + \varepsilon)^2 (3 - \beta_1 a_k)^2 - b_k^2]},$$

则 $\theta_k \in (\arctan[x_{k1}(\varepsilon)], \arctan[x_{k2}(\varepsilon)])$. 因此, 如果存在 $\varepsilon > 0$, 使得

$$\max_{k=1,2,\dots,n} \arctan[x_{k1}(\varepsilon)] \leq \min_{k=1,2,\dots,n} \arctan[x_{k2}(\varepsilon)],$$

那么存在 $\theta \in \left(\max_{k=1,2,\dots,n} \arctan[x_{k1}(\varepsilon)], \min_{k=1,2,\dots,n} \arctan[x_{k2}(\varepsilon)] \right)$, 使得 $\gamma_k(\theta) < 1$.

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Persistency of tripartite nonlocality sharing with noise^{*}

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Abstract

Recently, researchers have proven that an infinite number of Charlies and a pair of Alice and Bob can share standard tripartite nonlocality and genuinely nonsignal nonlocality by violating the Mermin and NS inequalities within tripartite systems. This discovery undoubtedly provides new perspectives and potential in quantum information science. However, it should be noted that the above-mentioned conclusion is derived on the highly idealized assumption that the quantum system is perfect and free from external disturbances. In reality, the realization of this ideal state is a challenging proposition. As a fundamental aspect of quantum mechanics, the phenomenon of quantum entanglement is susceptible to the influence of external factors, such as noise, during its practical implementation. Additionally, the process of quantum measurement can introduce potential errors, which may potentially diminish or even negate the observed quantum nonlocality. In light of the above situation, we investigate whether it is possible to share the corresponding quantum nonlocality, despite the inevitable occurrence of noise and error. This paper aims to study and discuss the persistency of nonlocality in noisy three-qubit systems. Firstly, the sufficient conditions are provided for Alice and Bob to share standard tripartite nonlocality with any number of Charlies, even when measurements are noisy and the initial three-qubit system is in a maximally entangled state with noise. This finding indicates that certain standard tripartite nonlocality can persist under non-ideal conditions as long as certain conditions are met. Moreover, this article elucidates the necessary conditions for multiple independent Charlies to share genuinely nonsignal nonlocality with a pair of Alice and Bob in a non-ideal state. This implies that despite the presence of noise and errors, this type of genuinely nonsignal nonlocality can still be securely shared among multiple parties as long as specific conditions are met. This research provides a new theoretical basis for the security and feasibility of quantum communication. The comprehensive analysis presented in this paper offers insights into the behavior of triple quantum nonlocality under noiseless conditions.

Keywords: nonlocality, triple quantum, noises

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含噪声的三体量子非局域共享的持久性

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Persistence of tripartite nonlocality sharing with noise

Zhao Lin-Yang He Kan Zhang Yan-Fang

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